On the Ulam stability of $F(z) + F(2z) = 0$

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Abstract In this paper it is shown that the complex functional equation $F(z) + F(2z) = 0$, $z \in \Omega := \mathbb{C} \setminus (-\infty, 0]$, is stable in the strong sense of Ulam. It means that given an analytic and bounded function $f(z)$ on Ω satisfying $|f(z) + f(2z)| < \delta$, $z \in \Omega$, for some $\delta > 0$, there exists an analytic solution $F(z)$ of the above functional equation such that $|f(z) - F(z)| < K(\delta)$ on each compact A of Ω, where $K(\delta)$ is a positive real function that tends to 0 as $\delta \to 0$. This result is extended to analytic functions $f(z)$ on Ω satisfying $|f(z) + f(2z)| < \delta$, $z \in \Omega$, for some $\delta > 0$, not necessarily bounded on Ω.

Keywords Spaces of bounded analytic functions of one complex variable- Approximation- Functional equations in the complex plane

Mathematics Subject Classification 30H05 - 30E10- 30D05

1 Introduction

D.H. Hyers, in his memorable paper [9], wrote:

"In a recent talk before the Mathematics Club of the University of Wisconsin, Dr. S. Ulam proposed the following problem of the "stability" of the equation $f(x+y) = f(x)+f(y)$. Suppose $f(x)$ satisfies this equation only approximately. Then does there exist a linear function which $f(x)$ approximates?".

This is the prologue of a theory, called **Hyers-Ulam stability,** which has produced a lot of works since 1941, when it was published the aforementioned Hyers's article answering in the afirmative Ulam's question focused on the Cauchy functional equation. The **Hyers-Ulam stability** theory has evolved and it has been subject to generalizations, from the initial problem posed on Banach spaces (see for instance [9], [10], [13]) to the latest stability problems on quasi-Banach algebras (see for instance [5]). In this way the primitive concept of Hyers-Ulam stability has been extended to the notion of **generalized Ulam–Hyers stability** or **stability in Ulam–Hyers–Bourgin** sense (see [2], [3]). But without doubt **Hyers–Ulam–Rassias stability** concept [10] has been a relevant impulse for the Stability Theory due to Themistocles M. Rassias. Regarding the **generalized Ulam–Hyers stability**, recently (see [8]) we have demonstrated that the generalized complex functional equation

$$F(z) + F(2z) + ... + F(nz) = 0,\ n \geq 2, \tag{1.1}$$

satisfies that property. To do it we have used the Fixed Point Alternative Theorem of Diaz and Margolis [4] and the connections among the solutions of (1.1) and the zeros of the n^{th} partial sum, $\zeta_n(z) = \sum_{m=1}^{n} \frac{1}{m^z}$, of the series that defines the Riemann zeta function on $\Re z > 1$ (for details see [6, Chap. 3], [7, Chap. 13], [8] and [12]).

In the celebrated Hyers's paper [9, Th.1], given a function $f(x)$ satisfying the condition $\|f(x+y)-f(x)-f(y)\|<\delta$ for some $\delta>0$, he explicitly constructed an additive function $F(x)$ solution of the Cauchy equation satisfying $\|F(x)-f(x)\|\leq\delta$. Here Hyers considered that F, f are "transformations of E into E'", where E, E' are Banach spaces. This is known as *direct method.* This method, in words of Rassias (see [13, p. 265]), "is a powerful tool for studying the stability of several functional equations. It is often used to construct a solution of a given functional equation".

Cauchy's equation $f(x+y)=f(x)+f(y)$ was the functional equation on which Ulam focused the notion of stability and it was the equation chosen by Hyers to solve Ulam's question. That equation synthesizes the linearity. However the real functional equation

$$f(x)+f(2x)=0,\ x>0, \tag{1.2}$$

introduced in [11], is far away from that the linear property. Indeed, we can directly check that all the functions of the family (all them are of class $\mathcal{C}^\infty$ on the open real interval $(0,+\infty)$)

$$\{f(x)=a\cos(\mu\log x)+b\sin(\mu\log x),\ x>0:\ a,\ b\in\mathbb{R}\},$$

where μ is any real number of the form $\mu=\frac{(2n+1)\pi}{\log 2}$, $n\in\mathbb{Z}$, are solutions of the above functional equation. Now it is perfectly clear the strong oscillatory behavior, especially near 0, of that family of solutions of (1.2). Consequently the character of the above functional equation is far from that of Cauchy equation.

In the present paper we consider the complex version of $f(x)+f(2x)=0$, i.e. the functional equation

$$F(z)+F(2z)=0, \tag{1.3}$$

where z belongs to the complex domain $\Omega:=\mathbb{C}\setminus(-\infty,0]$. Our goal is to prove that (1.3) is Ulam stable in a strong sense that will be precised below. We will demonstrate it by the direct method. That is, we will construct an analytic solution on Ω of (1.3), say $F(z)$, from a given function $f(z)$ with $|f(z)+f(2z)|<\delta$ for all $z\in\Omega$, satisfying $|F(z)-f(z)|<K(\delta)$ for all $z\in A$. Here A is a compact of Ω and $K(\delta)$ is a function of δ that tends to 0 as δ does.

Finally, in order to find applications of our results, we will say that (1.2) jointly with the functional equation $f(x)+f(2x)+f(3x)=0$, $x>0$, were already used to model a process relative to the combustion of hydrogen in a car engine [12]. On the other hand the direct method used to find the solution to our stability problem, itself is an efficient tool for studying the stability of functional equations.

To facilitate the reading we settle the notation that we will use along the paper. We introduce the following classes of functions on the complex domain $\Omega:=\mathbb{C}\setminus(-\infty,0]$:

(1) $\mathcal{H}(\Omega)$ represents the class of all analytic functions on Ω that are solution of the functional equation (1.3).

(2) $\mathcal{H}^{\infty}(\Omega)$ is the class of the functions of $\mathcal{H}(\Omega)$ that are bounded on Ω.

(3) For a given $\delta > 0$, $\mathcal{H}_{\delta}(\Omega)$ denotes the class of all analytic functions on Ω that satisfy the δ-property for the functional equation $F(z) + F(2z) = 0$. That is, all analytic functions $f(z)$ on Ω satisfying

$$|f(z) + f(2z)| < \delta \text{ for any } z \in \Omega. \tag{1.4}$$

(4) For a given $\delta > 0$, $\mathcal{H}_{\delta}^{\infty}(\Omega)$ is the class of the functions of $\mathcal{H}_{\delta}(\Omega)$ that are bounded on Ω.

(5) $\mathcal{H}^{\infty,S}(\Omega)$ is the class of the functions of $\mathcal{H}(\Omega)$ that are bounded on any infinite sector

$$S_{\theta_1,\theta_2} = \{z \in \Omega : \theta_1 \leq \arg z \leq \theta_2\}, \ -\pi < \theta_1 \leq \theta_2 < \pi. \tag{1.5}$$

(6) For a given $\delta > 0$, $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$ designates the class of all functions of $\mathcal{H}_{\delta}(\Omega)$ that are bounded on any sector S_{θ_1,θ_2} of (1.5).

We will prove that the above classes of functions are nonvoid (see below the lemmas). Excluding the case where a class is, by definition, included in another, there are relations of inclusion among them that are immediate such as: $\mathcal{H}(\Omega) \subset \mathcal{H}_{\delta}(\Omega)$, $\mathcal{H}^{\infty,S}(\Omega) \subset \mathcal{H}_{\delta}^{\infty,S}(\Omega)$ for any $\delta > 0$. However the strict inclusion $\mathcal{H}_{\delta}^{\infty}(\Omega) \subset \mathcal{H}_{\delta}^{\infty,S}(\Omega)$ or the equality $\mathcal{H}(\Omega) = \mathcal{H}^{\infty,S}(\Omega)$ are non-trivial (see below Lemma 4).

2 The stability of $F(z) + F(2z) = 0$ for the class $\mathcal{H}_{\delta}^{\infty}(\Omega)$

Lemma 1 *The class $\mathcal{H}_{\delta}^{\infty}(\Omega)$ defined in (4) is nonvoid.*

Proof. Consider a power function of the form $F(z) = z^{\beta}$ where β is a zero of the second approximation, $\zeta_2(z) = 1 + \frac{1}{2^z}$, of the series $\sum_{m=1}^{\infty} \frac{1}{m^z}$, $\Re z > 1$, that defines the Riemann zeta function $\zeta(z)$. Here the power function is defined as usual, i.e., $z^{\beta} := e^{\beta \log z}$, $z \in \Omega$, where $\log z$ is the principal branch of the logaritm, so $\log z := \ln |z| + i \arg z$ with $\arg z$ denoting the principal argument of z and then $\arg z \in (-\pi, \pi)$ (see for instance [1, p. 11-12]). We claim that if β is a zero of $\zeta_2(z)$, then $F(z) = z^{\beta}$ is an analytic and bounded function on Ω that satisfies the functional equation (1.3). Indeed, assume $z \in \Omega$ then $2z$ is also in Ω. We have

$$z^{\beta} + (2z)^{\beta} = z^{\beta}(1 + 2^{\beta}) = 0,$$

because $\zeta_2(\beta) = 0$ if and only $\zeta_2(-\beta) = 0$. Therefore $F(z) = z^{\beta}$ is an analytic function on Ω that satisfies the functional equation (1.3), so the class $\mathcal{H}(\Omega)$ defined in (1) is nonempty. It remains to prove that $F(z) = z^{\beta}$ is bounded on Ω. To do it we note that all the zeros of the function $\zeta_2(z) = 1 + \frac{1}{2^z}$ are of the form $\left\{\beta = \frac{(2n+1)\pi}{\log 2} i,\, n \in \mathbb{Z}\right\}$, so β lies on the imaginary axis. Then, given β a zero of $\zeta_2(z)$, there is an integer n such that

$$\left|z^{\beta}\right| = \left|e^{\beta \log z}\right| = \left|e^{\beta \ln |z| + \beta i \arg z}\right| = e^{-\frac{(2n+1)\pi}{\log 2} \arg z}.$$

Therefore, noticing $\arg z \in (-\pi, \pi)$, the function $F(z) = z^\beta$ is bounded on Ω, so the class $\mathcal{H}^\infty(\Omega)$ defined in (2) is nonvoid.

Now consider the class of functions

$$\mathcal{F} := \left\{ F(z) := Bz^\beta \text{ with } B \in \mathbb{C};\ \beta = \frac{(2n+1)\pi}{\log 2} i,\ n \in \mathbb{Z};\ z \in \Omega \right\}.$$

Then, for a given $\delta > 0$, the functions of the family

$$\mathcal{F}_\delta := \{ f(z) := F(z) + \alpha \text{ with } F(z) \in \mathcal{F},\ \alpha \in \mathbb{C},\ |\alpha| < \delta/2 \}$$

are analytic, bounded on Ω and they have the δ-property (1.4) for the functional equation $F(z) + F(2z) = 0$. Therefore $\mathcal{F}_\delta$ is contained in $\mathcal{H}_\delta^\infty(\Omega)$, so this class defined in (4) is nonvoid and consequently the class $\mathcal{H}_\delta(\Omega)$ defined in (3) is too. The proof of the lemma is now completed. ■

Now we can define the notion of Ulam stability of the functional equation (1.3).

Definition 2 *Let δ be a positive number. We will say that the functional equation $F(z) + F(2z) = 0$, $z \in \Omega := \mathbb{C} \setminus (-\infty, 0]$, is Ulam stable on a class $\mathcal{U} \subset \mathcal{H}_\delta(\Omega)$ if for any function $f(z) \in \mathcal{U}$ there exists a function $F(z) \in \mathcal{H}(\Omega)$ such that on each compact $A \subset \Omega$ one has*

$$|F(z) - f(z)| < K(\delta) \text{ for all } z \in A, \tag{2.1}$$

where $K(\delta)$ is a positive real function depending on δ such that $K(\delta) \to 0$ when $\delta \to 0$.

Theorem 3 *The functional equation $F(z) + F(2z) = 0$, $z \in \Omega := \mathbb{C} \setminus (-\infty, 0]$, is Ulam stable on the class $\mathcal{H}_\delta^\infty(\Omega)$ defined in (4).*

Proof. Let $f(z)$ be a function of $\mathcal{H}_\delta^\infty(\Omega)$. It is clear that if $f(z)$ is the null function, since it is a solution of the equation (1.3), then the theorem trivially follows. Assume $f(z)$ is a constant, say $C \neq 0$. Then noticing $f(z)$ satisfies (1.4), necessarily $|C| < \delta/2$. Therefore the function $F(z)$ identically 0 is analytic on Ω, satisfies the functional equation (1.3) and, by taking $K(\delta) := \delta/2$, the condition (1.4) is fulfilled for any compact A of Ω. Consequently, for constant functions the theorem trivially follows. Hence let $f(z)$ be a non-constant function of $\mathcal{H}_\delta^\infty(\Omega)$. Define the functions

$$\eta_m(z) := f(2^{-m}z) + f(2^{-m+1}z),\ m \in \mathbb{Z},\ z \in \Omega. \tag{2.2}$$

By making $m = 1$ in (2.2) we have $f(z) = -f(2^{-1}z) + \eta_1(z)$, and reiterating it for $m = 2, 3, \ldots$ we can write

$$f(z) = (-1)^m f(2^{-m}z) + (-1)^{m-1}\eta_m(z) + (-1)^{m-2}\eta_{m-1}(z) + \ldots + \eta_1(z), \tag{2.3}$$

for any $z \in \Omega$ and all $m \geq 1$. Since $f(z)$ is analytic and bounded on Ω, the functions of the sequence $((-1)^m f(2^{-m}z))_m$ are too. Hence by Montel

Theorem [1, p.165], there exists a convergent subsequence $((-1)^{m_k} f(2^{-m_k} z))_k$ in the topology of uniform convergence on the compacts of Ω. Let us denote

$$F(z) := \lim_{k\to\infty} (-1)^{m_k} f(2^{-m_k} z). \tag{2.4}$$

Consider the formula (2.3) for the values m_k of the above subsequence and take the limit when $k \to \infty$. Then, since there exists the limit (2.4), it also exists $\lim_{k\to\infty} \sum_{j=1}^{m_k} (-1)^{j-1} \eta_j(z)$ in the uniform topology of the compacts of Ω, and it is equal to $f(z) - F(z)$. That is, the series $\sum_{j=1}^{\infty} (-1)^{j-1} \eta_j(z)$ converges in the uniform topology of the compacts of Ω and its sum satisfies

$$\sum_{j=1}^{\infty} (-1)^{j-1} \eta_j(z) = f(z) - F(z). \tag{2.5}$$

Therefore the general term of the series tends to 0 in the uniform topology of the compacts of Ω and then

$$\lim_{j\to\infty} \eta_j(z) = 0, \tag{2.6}$$

uniformly on the compacts of Ω. Now we claim that the function $F(z)$ is a solution of (1.3). Indeed, by (2.4), (2.2) and (2.6), we have

$$F(z) + F(2z) = \lim_{k\to\infty} (-1)^{m_k} f(2^{-m_k} z) + \lim_{k\to\infty} (-1)^{m_k} f(2^{-m_k+1} z) =$$

$$= \lim_{k\to\infty} (-1)^{m_k} \left(f(2^{-m_k} z) + f(2^{-m_k+1} z) \right) = \lim_{k\to\infty} (-1)^{m_k} \eta_{m_k}(z) = 0.$$

Consequently $F(z)$ satisfies the equation (1.3) and then the claim follows.

Consider again the series $\sum_{j=1}^{\infty} (-1)^{j-1} \eta_j(z)$. From its convergence, given a compact A of Ω and $0 < \epsilon < \delta$, there exists j_0 such that

$$\left| \sum_{j>j_0} (-1)^{j-1} \eta_j(z) \right| < \epsilon, \text{ for any } z \in A.$$

On the other hand, noticing (2.2) and taking into account that $f(z)$ satisfies (1.4), each term of that series is such that

$$\left| (-1)^{j-1} \eta_j(z) \right| = \left| \eta_j(z) \right| < \delta, \text{ for all integer } j, \text{ for any } z \in \Omega.$$

Therefore, from (2.5), we get

$$|F(z) - f(z)| \le \sum_{j=1}^{j_0} \left| (-1)^{j-1} \eta_j(z) \right| + \left| \sum_{j>j_0} (-1)^{j-1} \eta_j(z) \right| < j_0 \delta + \epsilon, \tag{2.7}$$

for any $z \in A$. Then, since $j_0 \delta + \epsilon < (j_0 + 1)\delta$ and $(j_0 + 1)\delta \to 0$ as $\delta \to 0$, by defining $K(\delta) := (j_0 + 1)\delta$ the inequality (2.7) means that (2.1) is fulfilled. This completes the proof. ■

Now our goal is to extend the validity of the previous theorem to a more large class of functions (bounded or unbounded on Ω) that of $\mathcal{H}_{\delta}^{\infty}(\Omega)$. In the next Section we will prove that the class that we are looking for will be $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$ defined in (6). Its existence, properties and relation with $\mathcal{H}_{\delta}^{\infty}(\Omega)$ will be justified .

3 The stability of $F(z)+F(2z)=0$ for the class $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$

In order to justify the extension of the validity of the above theorem to a more large class of functions we need to prove previously the following result.

Lemma 4 *The class $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$ defined in (6) contains strictly to the class $\mathcal{H}_{\delta}^{\infty}(\Omega)$.*

Proof. If a function is bounded on Ω in particular is bounded on any sector (1.5), so $\mathcal{H}_{\delta}^{\infty}(\Omega)\subset\mathcal{H}_{\delta}^{\infty,S}(\Omega)$. Hence it only remains to prove that the previous inclusion is strict. In [12, Th. 10] we explicitely constructed an analytic solution on Ω of the functional equation (1.3), say $F_{\wp}(z)$, unbounded on Ω, by means of the function $\wp$ of Weierstrass having the two primitive periods $\ln 2$ and $2\pi i$. Therefore $F_{\wp}(z)\in\mathcal{H}(\Omega)$ but $F_{\wp}(z)\notin\mathcal{H}^{\infty}(\Omega)$.

In [12, Prop. 9]) was shown that any analytic solution on Ω of the functional equation (1.3) is necessarily bounded on any sector S_{θ_1,θ_2} of (1.5). Hence we have $\mathcal{H}(\Omega)=\mathcal{H}^{\infty,S}(\Omega)$, defined in (5). This proves that $F_{\wp}(z)$ belongs to the class $\mathcal{H}^{\infty,S}(\Omega)$. Then, given $\delta>0$ the functions of the family

$$\mathcal{G}_{\delta}:=\{f(z):=F_{\wp}(z)+\alpha \text{ with } \alpha\in\mathbb{C},\ |\alpha|<\delta/2\}$$

are analytic on Ω and satisfy the condition (1.4). Hence $\mathcal{G}_{\delta}$ is contained in $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$. However, since $F_{\wp}(z)\notin\mathcal{H}^{\infty}(\Omega)$, any function $f(z)\in\mathcal{G}_{\delta}$ is unbounded on Ω, so $f(z)\notin\mathcal{H}_{\delta}^{\infty}(\Omega)$. This proves that the class $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$ is strictly more large than $\mathcal{H}_{\delta}^{\infty}(\Omega)$. ■

The next result extends the validity of Theorem 3 to the class $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$.

Theorem 5 *The functional equation $F(z)+F(2z)=0$, $z\in\Omega:=\mathbb{C}\setminus(-\infty,0]$, is Ulam stable on the class $\mathcal{H}_{\delta}^{\infty,S}(\Omega)$ defined in (6).*

Proof. For a given $\delta>0$, consider an arbitrary function $f(z)\in\mathcal{H}_{\delta}^{\infty,S}(\Omega)$. If $f(z)$ is constant, as we have seen in Theorem 3, the result trivially follows. Hence assume $f(z)$ is non-constant (observe that $f(z)$ could be unbounded on Ω). Define the functions

$$\eta_m(z):=f(2^{-m}z)+f(2^{-m+1}z),\ m\in\mathbb{Z},\ z\in\Omega. \tag{3.1}$$

By making $m=1$ in (3.1) we have $f(z)=-f(2^{-1}z)+\eta_1(z)$, and reiterating it for $m=2$, 3, ... we can write

$$f(z)=(-1)^m f(2^{-m}z)+(-1)^{m-1}\eta_m(z)+(-1)^{m-2}\eta_{m-1}(z)+...+\eta_1(z), \tag{3.2}$$

for any $z \in \Omega$ and all $m \geq 1$. Since $f(z)$ is analytic and bounded on any sector $S_{\theta_1,\theta_2}$of (1.5), the functions of the sequence $((-1)^m f(2^{-m}z))_m$ are too (observe that if z is in a sector S_{θ_1,θ_2}, then $2^{-m}z \in S_{\theta_1,\theta_2}$ for all $m \in \mathbb{Z}$). On the other hand, it is immediate to check that given any compact A of Ω there exists a sector S_{θ_1,θ_2} that contains to A. Hence the sequence $((-1)^m f(2^{-m}z))_m$ is uniformly bounded on the compacts of Ω. Then by Montel Theorem [1, p.165], there exists a convergent subsequence $((-1)^{m_k} f(2^{-m_k}z))_k$ in the topology of uniform convergence on the compacts of Ω. Therefore by denoting

$$F(z) := \lim_{k\to\infty} (-1)^{m_k} f(2^{-m_k}z)$$

and repeating verbatim the argument of Theorem 3, from (2.4) to (2.7), we have that $F(z)$ is an analytic solution on Ω of (1.3) such that on each compact $A \subset \Omega$ satisfies

$$|F(z) - f(z)| < K(\delta) \text{ for all } z \in A,$$

with $K(\delta) \to 0$ as $\delta \to 0$. This proves definitely the theorem. ■

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